Allometric scaling: blood volume derivation

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As written in the article, the total volume of blood in an organism is given by the following sum:

$$V_b = \sum_{k=0}^{N} V_k N_k \tag{1}$$

where V_k is the volume of a tube in the k^{th} level of the network and N_k is the number of tubes at that level. First, we can rewrite Eq. 1 as

$$V_b = \sum_{k=0}^{N} \pi r_k^2 l_k m^k \tag{2}$$

where we have replaced V_k with the volume of a cylinder with radius r_k and length l_k, and we have replaced $N_k = m^k$ as the number of tubes at the k^{th} level, since $N_k = \prod_{i=0}^{k} m_i = m^k$ since $m_i = m$ for every network level.

Now, we use the ratio between successive radii and lengths to write the quantities r_k and l_k in terms of β, γ, r_{cap}, and l_{cap}, all of which are invariant across organisms. Recalling that $\beta_k = r_{k+1}/r_k = \beta$ and $\gamma_k = l_{k+1}/l_k = \gamma$, we can write

$$\begin{aligned} r_k &= \beta^{-1} r_{k+1} = \beta^{-2} r_{k+2} = \ldots = \beta^{-n} r_{k+n} \\ l_k &= \gamma^{-1} l_{k+1} = \gamma^{-2} l_{k+2} = \ldots = \gamma^{-n} l_{k+n} \end{aligned}$$

if we set $n + k = N$, i.e. we carry out this recursive relationship all the way up to level N, the level of capillaries, we can write Eq. 2 as

$$V_b = \sum_{k=0}^{N} \pi \left(\beta^{k-N} r_{cap}\right)^2 \left(\gamma^{k-N} l_{cap}\right) m^k = \pi \beta^{-2N} \gamma^{-N} r_{cap}^2 l_{cap} \sum_{k=0}^{N} (\beta^2 \gamma m)^k$$

Rewriting $a = \pi \beta^{-2N} \gamma^{-N} r_{cap}^2 l_{cap}$ and $x = \beta^2 \gamma m \neq 1$, we have a geometric series

$$\begin{aligned} \sum_{k=0}^{N} a x^k &= a \frac{1 - x^{N+1}}{1 - x} \\ \rightarrow V_b &= \left(\pi r_{cap}^2 l_{cap}\right) \left(\beta^2 \gamma\right)^{-N} \frac{1 - (\beta^2 \gamma m)^{N+1}}{1 - \beta^2 \gamma m} \\ V_b &= V_{cap} \left(\beta^2 \gamma\right)^{-N} \frac{(\beta^2 \gamma m)^{N+1}}{\beta^2 \gamma m} \frac{(\beta^2 \gamma m)^{-(N+1)} - 1}{(\beta^2 \gamma m)^{-1} - 1} \\ V_b &= V_{cap} N_{cap} \frac{(\beta^2 \gamma m)^{-(N+1)} - 1}{(\beta^2 \gamma m)^{-1} - 1} \end{aligned}$$

where we have replaced $V_{cap} = \pi r_{cap}^2 l_{cap}$ and $N_{cap} = m^N$. If we then make an approximation using $\beta^2 \gamma m < 1$ and $N \gg 1$,

$$V_b \approx V_{cap} N_{cap} \frac{(\beta^2 \gamma m)^{-N}}{1 - \beta^2 \gamma m}$$
$$V_b \propto (\beta^2 \gamma)^{-N}$$

since N_{cap} cancels out and all other terms ($V_{cap},\ \beta,\ \gamma,\ m$) are constant across organisms.